

Degenerate parametric oscillation in quantum membrane optomechanicsMónica Benito,^{1,2} Carlos Sánchez Muñoz,^{2,3} and Carlos Navarrete-Benlloch²¹*Instituto de Ciencia de Materiales, CSIC, Cantoblanco, 28049 Madrid, Spain*²*Max Planck Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany*³*Física Teórica de la Materia Condensada, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

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The promise of innovative applications has triggered the development of many modern technologies capable of exploiting quantum effects. But in addition to future applications, such quantum technologies have already provided us with the possibility of accessing quantum-mechanical scenarios that seemed unreachable just a few decades ago. With this spirit, in this work we show that modern optomechanical setups are mature enough to implement one of the most elusive models in the field of open system dynamics: degenerate parametric oscillation. Introduced in the eighties and motivated by its alleged implementability in nonlinear optical resonators, it rapidly became a paradigm for the study of dissipative phase transitions whose corresponding spontaneously broken symmetry is discrete. However, it was found that the intrinsic multimode nature of optical cavities makes it impossible to experimentally study the model all the way through its phase transition. In contrast, here we show that this long-awaited model can be implemented in the motion of a mechanical object dispersively coupled to the light contained in a cavity, when the latter is properly driven with multichromatic laser light. We focus on membranes as the mechanical element, showing that the main signatures of the degenerate parametric oscillation model can be studied in state-of-the-art setups, thus opening the possibility of analyzing spontaneous symmetry breaking and enhanced metrology in one of the cleanest dissipative phase transitions. In addition, the ideas put forward in this work would allow for the dissipative preparation of squeezed mechanical states.

DOI: [10.1103/PhysRevA.93.023846](https://doi.org/10.1103/PhysRevA.93.023846)**I. INTRODUCTION**

The last decades have seen the birth of a plethora of new technologies working in the quantum regime, starting with the laser [1–8], and including nonlinear optics [9–13], trapped ions [14–18] and atoms [19–27], cavity quantum electrodynamics [28–31], or, more recently, superconducting circuits [32–36] and optomechanical resonators [37–39]. Apart from their potential for quantum computation [40–42] and simulation [43–50], quantum metrology [51,52], and quantum communication [53–55], all these technologies have allowed us to reach physical scenarios that were nothing but a dream (or a “gedanken” experiment) for the founding fathers of quantum mechanics.

In this work we keep going deeper into the possibility of using new technologies to access phenomena predicted decades ago, but which have eluded observation so far, or only until very recently [56]. In particular, we show how modern optomechanical setups based on oscillating membranes [57–67] allow for the implementation of *degenerate parametric oscillation* (DPO), a fundamental model in the field of dissipative phase transitions [68–72]. Together with the laser, DPO is possibly the best-studied quantum-optical dissipative model, since it holds the paradigm of a phase transition whose associated spontaneously broken symmetry is discrete [71,72] (in contrast to those of the laser [73,74] or the nondegenerate parametric oscillator [12,75,76], which are continuous). Even though the main motivation for studying such a model came during the eighties from the possibility of implementing it with nonlinear optics [12,76] (see Fig. 1 and the next section), the intrinsic multimode nature of optical cavities prevents crossing the phase transition without additional locking techniques that break the symmetry of the system. In other words, despite the great deal of work invested in this model and its optical implementation, continuous-wave degenerate optical parametric oscillators (DOPOs) do not exist in reality.

The situation is rather different in the microwave realm of electronic circuits, where one can build single-mode cavities in the form of simple LC circuits. Indeed, it is in this context where DPO has been traditionally studied [77–79]. However, the strong thermal microwave background completely masks quantum-mechanical effects. This scenario was radically changed with the advent of superconducting circuits [32–36], which are cooled down to mK temperatures. This allowed, just a few months ago, to finally observe quantum-mechanical effects appearing above the DPO phase transition [56].

Apart from being a clean system where studying fundamental questions related to spontaneous symmetry breaking and ergodicity of open quantum systems [80–83], DPO might serve as a perfect test bed for enhanced metrology via dissipative phase transitions [84–87]. Motivated then by the interest that this model generates in different communities ranging from the purely theoretical to the most applied ones, in this work we show that DPO can be implemented in the motion of a membrane dispersively coupled to the field of an optical cavity [57–67], when a multichromatic laser properly drives the latter. Starting from a first-principles model, analytical and numerical methods allow us to identify the regimes where the desired model appears, as well as proving the feasibility of the scheme for the parameters under which current experiments take place. Our results open then the way to the experimental analysis of DPO in a completely different scenario, providing a feasible alternative to the recent circuit QED implementation.

II. DEGENERATE PARAMETRIC OSCILLATION AND ITS OPTICAL IMPLEMENTATION

In its minimal formulation (see Fig. 1) a DOPO consists of an optical cavity containing a crystal with second-order nonlinearity and pumped by a laser at frequency $2\omega_0$ for which

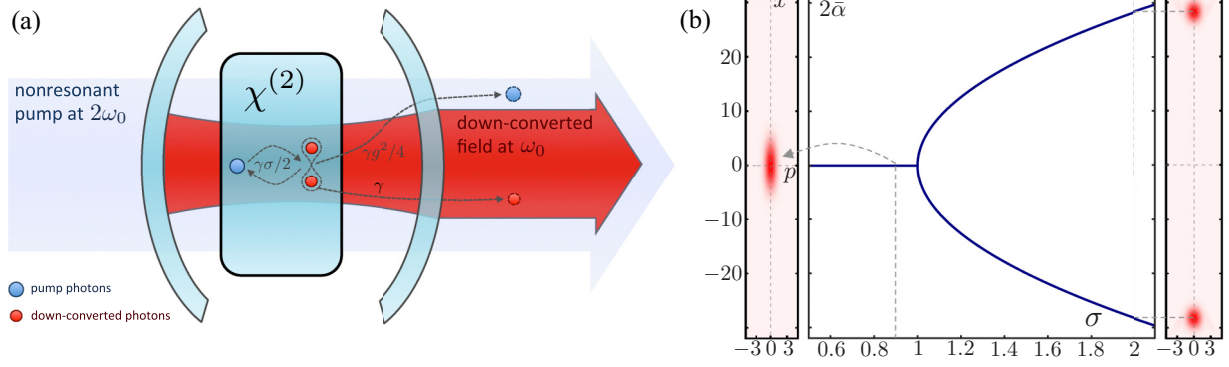


FIG. 1. (a) Sketch of the optical implementation of the DPO model. (b) Phase transition of the DPO model for $g = 0.1$. The central panel shows the steady-state field amplitude as a function of σ , in the classical limit. The left and right panels show the steady-state Wigner function for $\sigma = 0.9$ and 2 , respectively, which for our purposes can be interpreted as the joint probability density function $W(x, p)$ providing the statistics of “position” ($\hat{x} = \hat{a}^\dagger + \hat{a}$) and “momentum” ($\hat{p} = i\hat{a}^\dagger - i\hat{a}$) measurements.

the cavity is transparent (nonresonant-pump configuration). The parametric down-conversion process occurring inside the $\chi^{(2)}$ crystal is able to generate photons at the subharmonic frequency ω_0 , which is assumed resonant. The model can then be formulated as a master equation for the state $\hat{\rho}$ of the intracavity field [88,89]:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_{\text{DPO}}, \hat{\rho}] + \frac{\gamma g^2}{4} \mathcal{D}_{a^2}[\hat{\rho}] + \gamma \mathcal{D}_a[\hat{\rho}], \quad (1)$$

with

$$\hat{H}_{\text{DPO}} = \omega_0 \hat{a}^\dagger \hat{a} + i\gamma\sigma (e^{-2i\omega_0 t} \hat{a}^{\dagger 2} - e^{2i\omega_0 t} \hat{a}^2)/2, \quad (2)$$

where \hat{a} is the annihilation operator of cavity photons and we use the notation $\mathcal{D}_J[\hat{\rho}] = 2\hat{J}\hat{\rho}\hat{J}^\dagger - \hat{J}^\dagger\hat{J}\hat{\rho} - \hat{\rho}\hat{J}^\dagger\hat{J}$. The last term describes the loss of cavity photons through the partially transmitting mirror (with damping rate γ proportional to the mirror transmittance). The second term describes the loss of photon pairs which are up-converted to a pump photon that leaves the cavity (at rate $\gamma g^2/4$, where g is proportional to the crystal’s nonlinear susceptibility). Finally, the Hamiltonian term describes the exchange of photon pairs with the coherent background of the pumping field (at rate $\gamma\sigma/2$, where σ is proportional to the amplitude of the laser), as well as the free evolution of the cavity mode.

This master equation is invariant under the parity transformation $\hat{U} = (-1)^{\hat{a}^\dagger \hat{a}}$, which performs the operation $\hat{U}^\dagger \hat{a} \hat{U} = -\hat{a}$. On the other hand, defining $\bar{\alpha} = \lim_{t \rightarrow \infty} e^{i\omega_0 t} \langle \hat{a}(t) \rangle$, it is well known [12] that the classical limit of this equation predicts an *off* (or *below-threshold*) stationary state $\bar{\alpha} = 0$ for $\sigma \leq 1$, and an *on* (or *above-threshold*) phase-bistable state $\bar{\alpha} = \pm\sqrt{2(\sigma - 1)}/g$ for $\sigma > 1$; see Appendix A. Hence, at $\sigma = 1$ (*threshold*) the classical theory predicts a phase transition, accompanied by spontaneous symmetry breaking of the discrete phase above threshold, since the system has to choose between two possible steady states which individually do not preserve the symmetry.

In contrast to the classical state, the quantum steady-state solution $\bar{\rho} = \lim_{t \rightarrow \infty} e^{i\omega_0 t \hat{a}^\dagger \hat{a}} \hat{\rho}(t) e^{-i\omega_0 t \hat{a}^\dagger \hat{a}}$ of Eq. (1) is unique for any σ [72,89,90]. The symmetry of the master equation, together with the uniqueness of the steady state, forces the latter to be invariant under the transformation as well, $\hat{U} \bar{\rho} \hat{U}^\dagger = \bar{\rho}$,

which in turn implies that $\bar{\alpha} = \text{tr}\{\bar{\rho}\hat{a}\} = 0 \forall \sigma$, which seems to contradict the classical phase-bistability prediction. However, the situation is a bit more subtle: as shown in Fig. 1(b) through the Wigner function [91,92] (see also Appendix A), below threshold the quantum state is a squeezed state centered at the origin of phase space, while above threshold it develops two lobes centered (approximately) around the classical solutions. Hence, quantum mechanically, the classical phase transition has the significance of a crossover between phases which preserve the symmetry in two physically distinct ways.

Unfortunately, in real experiments, degenerate down-conversion has to compete with the generation of photon pairs at nondegenerate frequencies ω_1 and ω_2 such that $\omega_1 + \omega_2 = 2\omega_0$ (*energy conservation*), and phase-matching in the nonlinear crystal (*momentum conservation*) tends to give preference to one of such processes above threshold [93–95]. Even though modern devices are able to produce photon pairs oscillating even within the same longitudinal cavity mode [96], exact frequency degeneracy cannot be ensured without additional locking techniques that compromise the phase transition, since, in the words of Ref. [97], “degeneracy occurs only accidentally since it corresponds to a single point in the experimental parameter space.” Therefore, these devices cannot be used to study experimentally the DPO model all the way through its phase transition.

III. OPTOMECHANICAL IMPLEMENTATION OF DEGENERATE PARAMETRIC OSCILLATION

In contrast to the optical case, we show that optomechanical resonators in which a mechanical degree of freedom is dispersively coupled to the cavity field allow for the study of the full DPO model. Our proposal follows closely current experimental setups based on dielectric membranes [57–67]. In such setups, the type of coupling arising between the membrane’s motion and a given driven cavity mode depends on the position of the former with respect to the standing wave defined by the latter [57,58,65]. In particular, denoting by \hat{x} the displacement of the membrane with respect to its equilibrium position (normalized to its zero-point fluctuations [37]), the frequency

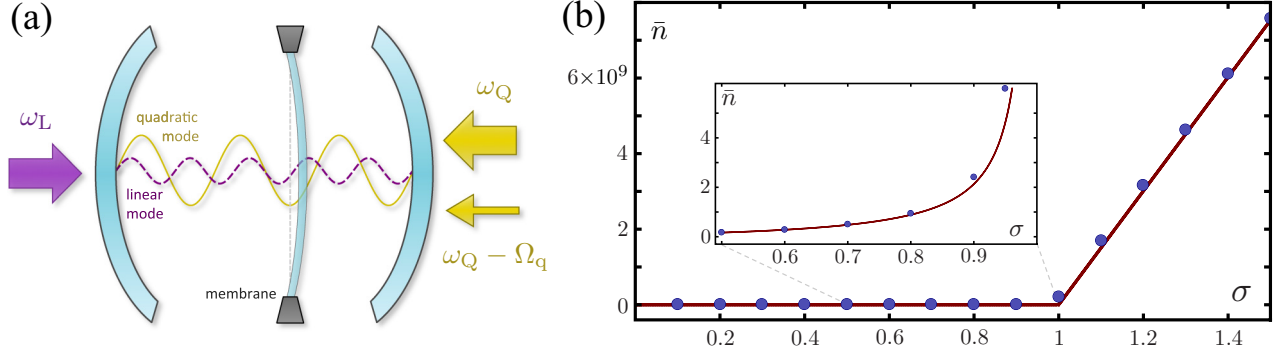


FIG. 2. (a) Sketch of the optomechanical setup with which we propose to implement the DPO model. (b) Asymptotic phonon number as a function of the pump parameter, for a set of parameters in correspondence with current experiments (see the text for details). The main plot corresponds to the predictions in the classical limit, and the inset corresponds to the predictions obtained from the asymptotic state generated by the master equation. The blue circles correspond to the optomechanical model, while the red curves are evaluated from its effective DPO model. Note the good agreement between both models.

shift felt by the optical mode is proportional to \hat{x}^2 when the membrane is located in a node or an antinode, while it is proportional to \hat{x} when it is halfway between them. In most optomechanical systems the linear coupling dominates, and many exciting phenomena have been already proven by exploiting it, including mechanical cooling [57,60,62,67,98–106], optical squeezing [107–110], and induced transparency [66,111–114]. On the other hand, the quadratic coupling has already promised very interesting applications such as quantum nondemolition measurements of the phonon number [57,58], and in our case it will provide the leading mechanism to achieve DPO.

Our proposal is sketched in Fig. 2(a). We consider two optical modes with frequencies ω_l and ω_q , linearly and quadratically coupled to the fundamental mechanical mode of the membrane (with frequency Ω), respectively; consequently, we will refer to them as the *linear* and *quadratic* modes. In the absence of optomechanical coupling, the membrane is at some equilibrium temperature T with its thermal environment, which drives it at some rate γ_m to a thermal state with mean phonon number $\bar{n}_{th} = k_B T / \hbar \Omega \gg 1$. However, we assume the linear mode to be driven by a monochromatic laser at frequency ω_L tuned to cool down the membrane to an effective phonon number $\bar{n}_{eff} \approx \bar{n}_{th} / C_1$ at rate $\gamma_{eff} = C_1 \gamma_m$, where C_1 is the cooperativity of the linear optomechanical coupling [37]; current experiments reach cooperativities $\sim 10\,000$, which together with cryogenic temperatures allow us to cool down the mechanical motion close to its ground state [67] $\bar{n}_{eff} = 0$, which we assume in the following. On the other hand, the quadratic mode is driven with a laser containing a tone at frequency ω_Q , plus a sideband at frequency $\omega_Q - \Omega_q$. We will use the first tone to create the two-phonon losses \mathcal{D}_{a^2} needed in the DPO model (1), while the combined action of the two tones will provide the coherent exchange of phonon pairs present in the Hamiltonian (2).

We model the system by a master equation for its state $\hat{\rho}$, which in a frame rotating at the laser frequency ω_Q takes the form

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_{qm} + \hat{H}_q(t), \hat{\rho}] + \gamma_q \mathcal{D}_{a_q}[\hat{\rho}] + \gamma_{eff} \mathcal{D}_b[\hat{\rho}], \quad (3)$$

with Hamiltonian terms

$$\hat{H}_{qm} = \Omega \hat{b}^\dagger \hat{b} - g_q \hat{a}_q^\dagger \hat{a}_q \hat{x}^2, \quad (4a)$$

$$\hat{H}_q(t) = -\Delta_q \hat{a}_q^\dagger \hat{a}_q + i[\mathcal{E}_q(t) \hat{a}_q^\dagger - \mathcal{E}_q^*(t) \hat{a}_q]. \quad (4b)$$

The bichromatic driving amplitude can be written as $\mathcal{E}_q(t) = \mathcal{E}_0 + \mathcal{E}_1 e^{-i\Omega_q t + i\phi}$, with ϕ some relative phase between the two tones. $\Delta_q = \omega_Q - \omega_q$ is the laser detuning. \hat{b} is the mechanical annihilation operator, from which the mechanical displacement is written as $\hat{x} = \hat{b} + \hat{b}^\dagger$. \hat{a}_q is the quadratic mode's annihilation operator, with corresponding cavity damping rate γ_q . The driving amplitudes $\mathcal{E}_{0,1}$ are written in terms of the power $P_{0,1}$ of the laser at the corresponding frequency as $\mathcal{E}_{0,1} \approx \sqrt{2\gamma_q P_{0,1}} / \hbar \omega_q$ [76].

We can understand the conditions under which this bichromatically driven optomechanical setup is mapped to the DPO model by adiabatically eliminating the quadratic mode. We provide the details of the derivation in Appendix B, and here we just point out some relevant steps. We follow the usual projector-superoperator technique [115–118] in which the quadratic mode is assumed to be in some reference state and follow some reference dynamics. For our current purposes, it is enough to assume that it does not feel any mechanical backaction, so that its dynamics is described by the master equation above with $g_q = 0$. This means that (i) we can take a coherent state with amplitude

$$\alpha_q(t) = e^{i \arctan(\Delta_q / \gamma_q)} [\sqrt{\bar{n}_0} - i \sqrt{\bar{n}_1} e^{-i\Omega_q t}], \quad (5)$$

as its reference state, where we have made a concrete choice of the second tone's phase ϕ that simplifies the expression [see Eq. (B5)] and defined

$$\bar{n}_j = \mathcal{E}_j^2 / [\gamma_q^2 + (\Delta_q + j\Omega_q)^2], \quad (6)$$

which are the number of photons introduced by the corresponding laser in the cavity; and (ii) all the correlation functions of its quantum fluctuations $\delta \hat{a}_q = \hat{a}_q - \alpha_q$ will decay in time at rate γ_q .

Keeping in mind the traditional picture of sideband cooling [37], it is intuitive to understand how the DPO model arises from \hat{H}_{qm} upon adiabatic elimination of the optical mode.

First, it is convenient to work in the weak-sideband regime $\bar{n}_1 \ll \bar{n}_0$ in order to avoid undesired time-dependent terms from being non-negligible (see Appendix B). The coherent part of the optical field generates then an effective mechanical Hamiltonian

$$\begin{aligned} \hat{H}_{\text{eff}} &= \Omega \hat{b}^\dagger \hat{b} - g_q |\alpha_q(t)|^2 \hat{x}^2 \\ &\approx \Omega_{\text{eff}} \hat{b}^\dagger \hat{b} + i g_q \sqrt{\bar{n}_0 \bar{n}_1} (e^{i\Omega_q t} \hat{b}^2 - \text{H.c.}), \end{aligned} \quad (7)$$

where $\Omega_{\text{eff}} = \Omega - 2g_q \bar{n}_0$, and any other term can be neglected as long as we work within the rotating-wave approximation $\Omega_{\text{eff}} \gg g_q \bar{n}_0$; see Appendix B for details. Therefore, choosing the second tone as $\Omega_q = 2\Omega_{\text{eff}}$, this effective Hamiltonian provides precisely \hat{H}_{DPO} ; see Eq. (1).

On the other hand, the elimination of the optical fluctuations generates both dissipative and Hamiltonian mechanical terms. Within the weak sideband and rotating-wave approximations, all the Hamiltonian terms can be neglected, while only the dissipators \mathcal{D}_{b^2} , $\mathcal{D}_{b^{\dagger 2}}$, and $\mathcal{D}_{b^\dagger b}$ survive, corresponding, respectively, to two-phonon cooling, heating, and dephasing. In the weak-sideband regime, the rate of each process is static and solely controlled by the fundamental tone. Hence, setting its detuning to the red two-phonon resonance, $\Delta_q = -2\Omega_{\text{eff}}$, while working in the resolved sideband regime, $4\Omega_{\text{eff}}^2 \gg \gamma_q^2$, the heating and dephasing terms are highly suppressed just as in standard cooling [37], leaving us with a two-photon cooling dissipator at rate $\gamma_m C_q$, where we have introduced the cooperativity $C_q = g_q^2 \bar{n}_0 / \gamma_q \gamma_m$. It is important to note that the Markov approximation (central to this method), requires a decay of the optical correlators much faster than the effective mechanical dynamics induced by the optical fields (see Appendix B), that is, $\gamma_q \gg \max\{\gamma_{\text{eff}}, \gamma_m C_q, g_q \sqrt{\bar{n}_0 \bar{n}_1}\}$.

This shows that the dynamics of the mechanical mode should follow an effective DPO master equation of the form (1), with $\omega_0 = \Omega_{\text{eff}}$, $\gamma = \gamma_{\text{eff}}$, $g^2 = 4C_q / C_1$, and $\sigma = g \sqrt{\bar{n}_1 \gamma_q / \gamma_{\text{eff}}}$. Remarkably, we obtain a set of parameters that can be optically tuned by means of the two laser powers P_0 and P_1 and that, as we show next, allow us to explore physical regimes not available in all-optical implementations.

IV. IMPLEMENTABILITY IN CURRENT SETUPS

We take the experiments of Ref. [67] as a reference, in which $\Omega = 4.4$ MHz, $\gamma_m = 0.8$ Hz, $\omega_q = 1770$ THz, $\gamma_q = 1.3$ MHz, $C_1 = 10\,000$, and $g_q = 10^{-5}$ Hz (this last parameter taken from Ref. [65]). In order to stay safely within the rotating-wave approximation we must impose a bound on the intracavity photon number given by $\bar{n}_0 \ll \Omega / 2g_q \approx 10^{11}$, which also certifies that we are within the resolved sideband regime, $\Omega_{\text{eff}} / \gamma_q \approx 3.5$. Taking $\bar{n}_0 = 3.3 \times 10^9$ (corresponding to a power of $P_0 \approx 20$ mW, fairly reasonable in such setups), we then obtain a quadratic cooperativity $C_q \approx 3.3 \times 10^{-7}$, leading to an effective two-phonon loss $g \approx 10^{-5}$, on the order of the one obtained in optical implementations [76]. On the other hand, let us take $\bar{n}_1 = 3 \times 10^8 \ll \bar{n}_0$ (corresponding to $P_1 \approx 35$ μ W) as an upper bound for the sideband photon number; then by varying the sideband power from zero to this value, the effective σ parameter can be varied all the way through the phase transition and up to $\sigma = 2.5$, showing how current optomechanical setups should be able to reach

regions of the DPO model beyond what is possible in optical implementations. Note finally that the Markov approximation is very well satisfied since $\gamma_q / \gamma_{\text{eff}} \approx 150$.

V. NUMERICAL SIMULATIONS

In order to certify the predictions offered above with the effective mechanical model under optical adiabatic elimination, we have performed numerical simulations of the full optomechanical problem for the realistic parameters of the previous section.

We proceed in two ways. First, by directly simulating the evolution induced by the master equation (3) in a truncated Fock basis [119]. We are interested in the asymptotic phonon number $\lim_{t \rightarrow \infty} \langle \hat{b}^\dagger \hat{b}(t) \rangle$, which oscillates at frequency Ω_q and can be approximately written as $\bar{n} + \delta n \sin(\Omega_q t)$, where typically $\bar{n} \gg \delta n$. In the inset of Fig. 2(b) we show the static phonon background \bar{n} as a function of the effective σ (which we remark that can be tuned through the sideband power P_1 in an experiment), together with the phonon number predicted from the effective DPO master equation (1). We find a very good agreement between these quantities, up to the value of σ that we have been able to simulate. For large phonon numbers (for $\sigma > 0.95$ in our case) this type of “brute force” simulation becomes unfeasible, as the Fock-space truncation needed to capture the state becomes too large.

Fortunately, in order to prove that the DPO phase transition is present in the full model, it is enough to consider the classical limit, which is the second type of simulation that we have performed. In this limit the optical and mechanical states are assumed to be coherent, hence described by complex amplitudes α and β , respectively, which, defining the mechanical position $x = 2\text{Re}\{\beta\}$ and momentum $p = 2\text{Im}\{\beta\}$, are shown in Appendix C to evolve according to

$$\dot{x} = \Omega p, \quad (8a)$$

$$\dot{p} = -2\gamma_{\text{eff}} p - (\Omega - 4g_q |\alpha|^2) x, \quad (8b)$$

$$\dot{\alpha} = \mathcal{E}_q(t) - (\gamma_q - i\Delta_q - i g_q x^2) \alpha. \quad (8c)$$

These are a set of coupled nonlinear equations which can be efficiently simulated in virtually all parameter space. Their prediction for \bar{n} is plotted in Fig. 2(b) as a function of σ , where we also show the predictions of the DPO model, given by $\bar{n} = 2(\sigma - 1)/g^2$. We see that this classical simulation also finds very good agreement with the DPO model, in particular showing the phase transition exactly as expected.

VI. CONCLUSIONS

In summary, we have shown that the elusive degenerate parametric oscillator model can be realistically implemented in current optomechanical setups. Apart from providing the possibility of studying experimentally many interesting theoretical predictions put forward during the last three decades, the implementation of this simple (but paradigmatic) dissipative model in modern quantum technologies opens the way to analyzing open questions related to ergodicity and spontaneous symmetry breaking, as well as enhanced metrology with dissipative phase transitions. In addition, it would allow for the dissipative preparation of squeezed mechanical states, which

are receiving a lot of attention lately [120–122]. Let us finally remark that even though we have focused on a membrane-based implementation, our ideas can be realized with any other optomechanical system allowing for a quadratic coupling to the mechanical position, including double-microdisk resonators [123], microdisk-nanocantilever [124] or microspher-nanostring [125] devices, paddle nanocavities [126], and photonic crystals [127].

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APPENDIX A: ASYMPTOTIC STATE OF DEGENERATE PARAMETRIC OSCILLATOR

In this Appendix we provide further details about the way in which we found the asymptotic state of the DPO model, both in the quantum and classical regimes.

1. Moving to a simpler picture

The asymptotic state of the DPO model (1) is better analyzed in a picture rotating at frequency ω_0 , where the state becomes time independent. In particular, defining the transformation operator $\hat{U}_c = \exp(i\omega_0 t \hat{a}^\dagger \hat{a})$, in the new picture the state of the system $\tilde{\rho} = \hat{U}_c \hat{\rho} \hat{U}_c^\dagger$ obeys the master equation

$$\frac{d\tilde{\rho}}{dt} = -i[\tilde{H}_{\text{DPO}}, \tilde{\rho}] + \frac{\gamma g^2}{4} \mathcal{D}_{a^2}[\tilde{\rho}] + \gamma \mathcal{D}_a[\tilde{\rho}], \quad (\text{A1})$$

where the transformed Hamiltonian reads

$$\tilde{H}_{\text{DPO}} = \hat{U}_c \hat{H}_{\text{DPO}} \hat{U}_c^\dagger - \omega_0 \hat{a}^\dagger \hat{a} = i\gamma\sigma(\hat{a}^{\dagger 2} - \hat{a}^2)/2. \quad (\text{A2})$$

Hence, we see how in this picture the master equation becomes time independent, leading to a stationary asymptotic state of the system.

2. Quantum asymptotic state

Indeed, the unique steady state of this master equation is known analytically, in particular in the form of a positive P distribution [90], from which in principle the elements of the density operator can be reconstructed in any basis [128]. However, such a reconstruction is computationally very demanding, and for our purposes it is simpler to evaluate the steady state numerically. Concretely, in Sec. II we show the

Wigner function associated with the steady state for $g = 0.1$ and two different values of the pump parameter, $\sigma = 0.9$ and $\sigma = 2$, above and below the phase transition, respectively. Let us now spend some lines explaining how we have computed this steady states and their corresponding Wigner functions.

As explained in detail in Ref. [119], we perform the numerics by going to superspace, where the elements of the density matrix in the Fock basis are gathered in a vector $\vec{\rho}$, and the master equation becomes then a linear system $d\vec{\rho}/dt = \mathbb{L}_{\text{DPO}}\vec{\rho}$, where \mathbb{L}_{DPO} is a representation of the Liouville superoperator which induces the DPO dynamics, $\mathcal{L}_{\text{DPO}}[\cdot] = -i[\tilde{H}_{\text{DPO}}, \cdot] + \gamma \mathcal{D}_a[\cdot] + \gamma g^2 \mathcal{D}_{a^2}[\cdot]/4$. Note that the Fock basis is infinite dimensional, and hence one has to introduce a truncation $\{|n\rangle\}_{n=0,1,\dots,N}$ in order to work in the computer. In our case, we follow the criterion of truncating to values of N for which the observables we are interested in (e.g., the photon number $\langle \hat{a}^\dagger \hat{a} \rangle$) converge up to a three-digit precision or more. The steady state corresponds then to the eigenvector with zero eigenvalue of \mathbb{L}_{DPO} , which provides us with the Fock basis components of the steady-state operator, $\{\tilde{\rho}_{mn}\}_{m,n=0,1,\dots,N}$.

Once we have the density operator in the Fock basis, the Wigner function can be evaluated as follows: First, given a harmonic oscillator with position and momentum quadratures, $\hat{x} = \hat{a}^\dagger + \hat{a}$ and $\hat{p} = i(\hat{a}^\dagger - \hat{a})$, respectively, recall that the Wigner function $W(x, p)$ can be seen as a joint probability density function for measurements of such observables, in the sense that the marginal $P(x) = \int_{\mathbb{R}} dp W(x, p)$ provides the probability density function predicting the statistics of position measurements (and similarly for the momentum). Defining the polar coordinates (r, φ) in phase space by $(x, p) = r(\cos \varphi, \sin \varphi)$, the Wigner function of the steady state $\tilde{\rho}$ can be found from its components in the Fock basis as [129–132]

$$\bar{W}(r, \varphi) = \sum_{mn=0}^N \tilde{\rho}_{mn} W_{mn}(r, \varphi), \quad (\text{A3})$$

where we have defined the Wigner function of the operator $|m\rangle\langle n|$, given by

$$W_{mn}(r, \varphi) = \frac{(-1)^n}{\pi} \sqrt{\frac{n!}{m!}} e^{i\varphi(m-n)} r^{m-n} L_n^{m-n}(r^2) e^{-r^2/2}, \quad (\text{A4})$$

with $L_n^p(x)$ the modified Laguerre polynomials and where we have assumed $m \geq n$ (note that $W_{nm} = W_{mn}^*$).

3. Classical limit and steady state

Phase transitions in dissipative systems are usually revealed in the classical limit of the corresponding quantum models. Let us explain how such a limit can be obtained from the master equation that describes the system quantum mechanically. The idea is indeed quite simple in the case of bosonic systems: the classical limit consists of assuming that the state of all bosonic modes is coherent, with an amplitude that will play the role of the classical variable.

In particular, in the case of the single-bosonic mode considered in the DPO model, this means that we assume its state to be a coherent state $|\alpha(t)\rangle$ at all times, such that the expectation value of any normally ordered observable factorizes as $\langle \hat{a}^{\dagger k}(t) \hat{a}^l(t) \rangle = \alpha^{*k}(t) \alpha^l(t)$, where for convenience we are defining expectation values with respect to the state in the

picture rotating at the laser frequency, that is $\langle \hat{a}^{\dagger k}(t) \hat{a}^l(t) \rangle = \text{tr}\{\hat{a}^{\dagger k} \hat{a}^l \tilde{\rho}(t)\}$. This allows us to find an evolution equation for $\alpha(t)$ from the master equation (A1) as follows: given any operator \hat{A} , the master equation allows us to write the evolution of its expectation value as

$$\begin{aligned} \frac{d\langle \hat{A} \rangle}{dt} &= \text{tr} \left\{ \hat{A} \frac{d\tilde{\rho}}{dt} \right\} \\ &= -i\langle [\hat{A}, \hat{H}_{\text{DOPO}}] \rangle + \gamma \langle (\hat{a}^\dagger, \hat{A}) \hat{a} \rangle + \langle \hat{a}^\dagger [\hat{A}, \hat{a}] \rangle \\ &\quad + \frac{\gamma g^2}{4} \langle ([\hat{a}^{\dagger 2}, \hat{A}] \hat{a}^2) + \langle \hat{a}^{\dagger 2} [\hat{A}, \hat{a}^2] \rangle, \end{aligned} \quad (\text{A5})$$

which applied to the annihilation operator, and using the coherent-state ansatz, provides us with the classical equation of DPO:

$$\gamma^{-1} \dot{\alpha} = \sigma \alpha^* - \frac{g^2}{2} |\alpha|^2 \alpha - \alpha. \quad (\text{A6})$$

This is indeed the equation that would have been obtained by using classical electromagnetic theory on the DOPO, where α would be interpreted as the normalized amplitude of the optical field. As explained in the text, this equation has two types of asymptotic stationary ($\dot{\alpha} = 0$) solutions: a trivial one $\bar{\alpha} = 0$, and a nontrivial one $\bar{\alpha} = \pm \sqrt{2(\sigma - 1)}/g$, which exists only for $\sigma > 1$ and has sign-indeterminacy owed to the symmetry $\alpha \rightarrow -\alpha$ of Eq. (A6). We use the bar to denote ‘‘stationary state.’’ In order for these solutions to be physical, they need to be stable against perturbations; their stability can be analyzed by studying the evolution of small perturbations around them; that is, by writing $\alpha(t) = \bar{\alpha} + \delta\alpha(t)$, and linearizing Eq. (A6) with respect to $\delta\alpha$. Defining the vector $\delta\alpha = \text{col}(\delta\alpha, \delta\alpha^*)$, one obtains the linear system $\delta\dot{\alpha} = \mathcal{M}\delta\alpha$, where the linear stability matrix reads

$$\mathcal{M} = \begin{pmatrix} -1 - 2g^2|\bar{\alpha}|^2 & \sigma - g^2\bar{\alpha}^2/2 \\ \sigma - g^2\bar{\alpha}^{*2}/2 & -1 - 2g^2|\bar{\alpha}|^2 \end{pmatrix}. \quad (\text{A7})$$

Hence, the stability of a given stationary solution $\bar{\alpha}$ is determined by the eigenvalues of this matrix: when they all have negative real part, it will be stable, while if some of them have positive real part, perturbations will tend to grow, showing that the solution is unstable. In the case of the trivial solution, the eigenvalues are $\lambda_{\pm} = -1 \pm \sigma$, and hence, it is unstable for $\sigma > 1$. On the other hand, the eigenvalues associated with the nontrivial solution read $\lambda_{\pm} = -2\sigma + 1 \pm 1$, which are always negative for $\sigma > 1$, and hence this solution is stable.

Therefore, we see that, at the classical level, the phase transition is revealed by a nonanalytic change in the stationary solution at threshold $\sigma = 1$.

APPENDIX B: ADIABATIC ELIMINATION OF OPTICAL MODE

In this Appendix we perform a careful adiabatic elimination of the quadratic optical field, identifying the region of the parameter space where the system is expected to give rise to the DPO model. Note that our optomechanical master equation (3) cannot be made time independent in any picture and, hence, while we follow well-known projector superoperator techniques [115,116], we need to be careful and adapt them to our time-dependent problem.

1. Moving to a simpler picture

Our starting point is the master equation of the optomechanical system as we introduced it in the main text:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_m + \hat{H}_q(t) + \hat{H}_{\text{qm}}, \hat{\rho}] + \gamma_q \mathcal{D}_{a_q}[\hat{\rho}] + \gamma_{\text{eff}} \mathcal{D}_b[\hat{\rho}], \quad (\text{B1})$$

with Hamiltonian terms

$$\hat{H}_m = \Omega \hat{b}^\dagger \hat{b}, \quad (\text{B2a})$$

$$\hat{H}_q = -\Delta_q \hat{a}_q^\dagger \hat{a}_q + i[\mathcal{E}_q(t) \hat{a}_q^\dagger - \mathcal{E}_q^*(t) \hat{a}_q], \quad (\text{B2b})$$

$$\hat{H}_{\text{qm}} = -g_q \hat{a}_q^\dagger \hat{a}_q \hat{x}^2, \quad (\text{B2c})$$

and where the bichromatic driving amplitude of the quadratic mode can be written as

$$\mathcal{E}_q(t) = \mathcal{E}_0 + \mathcal{E}_1 e^{-i\Omega_q t + i\phi}, \quad (\text{B2d})$$

with ϕ some relative phase between the two tones that will be chosen shortly. All the symbols have the meaning introduced in the main text.

In order to perform the adiabatic elimination of the optical mode, it is convenient to move to a picture where the large coherent background that the driving fields create in the optical mode is already taken into account. This is accomplished by using a displacement $\hat{D}[\alpha_q(t)] = \exp[\alpha_q(t) \hat{a}_q^\dagger - \alpha_q^*(t) \hat{a}_q]$ as the transformation operator, where the amplitude $\alpha_q(t)$ is chosen to obey the evolution equation

$$\dot{\alpha}_q = \mathcal{E}_q(t) - (\gamma_q - i\Delta_q)\alpha_q, \quad (\text{B3})$$

with solution

$$\begin{aligned} \alpha_q(t) &= \alpha_q(0) e^{-(\gamma_q - i\Delta_q)t} + \frac{\mathcal{E}_0}{\gamma_q - i\Delta_q} [1 - e^{-(\gamma_q - i\Delta_q)t}] \\ &\quad + \frac{\mathcal{E}_1 e^{i\phi}}{\gamma_q - i(\Delta_q + \Omega_q)} [e^{-i\Omega_q t} - e^{-(\gamma_q - i\Delta_q)t}]. \end{aligned} \quad (\text{B4})$$

Note that, by choosing

$$\phi = -\pi/2 + \arctan(\Delta_q/\gamma_q) - \arctan[(\Delta_q + \Omega_q)/\gamma_q], \quad (\text{B5})$$

we obtain the asymptotic displacement

$$\lim_{t \gg \gamma_q^{-1}} \alpha_q(t) = e^{i\arctan(\Delta_q/\gamma_q)} [\sqrt{\bar{n}_0} - i\sqrt{\bar{n}_1} e^{-i\Omega_q t}], \quad (\text{B6})$$

with $\bar{n}_0 = \mathcal{E}_0^2/[\gamma_q^2 + \Delta_q^2]$ and $\bar{n}_1 = \mathcal{E}_1^2/[\gamma_q^2 + (\Delta_q + \Omega_q)^2]$, which is the amplitude that we introduced in the main text. In the following we assume to be working in this asymptotic regime $t \gg \gamma_q^{-1}$, even if we do not write the limit explicitly to shorten the expressions. In this new picture, the transformed state $\tilde{\rho} = \hat{D}^\dagger[\alpha_q(t)] \hat{\rho} \hat{D}[\alpha_q(t)]$ evolves then according to

$$\frac{d\tilde{\rho}}{dt} = -i[\tilde{H}(t), \tilde{\rho}] + \gamma_q \mathcal{D}_{a_q}[\tilde{\rho}] + \gamma_{\text{eff}} \mathcal{D}_b[\tilde{\rho}], \quad (\text{B7})$$

where the transformed Hamiltonian

$$\begin{aligned} \tilde{H}(t) &= \hat{D}^\dagger[\alpha_q(t)] [\hat{H}_m + \hat{H}_q(t) + \hat{H}_{\text{qm}}] \hat{D}[\alpha_q(t)] \\ &\quad + i(\dot{\alpha}_q^* \hat{a}_q - \dot{\alpha}_q \hat{a}_q^\dagger), \end{aligned} \quad (\text{B8})$$

can be written as the sum of three terms, $\tilde{H} = \tilde{H}_q + \tilde{H}_m(t) + \tilde{H}_{qm}(t)$, with

$$\tilde{H}_q = -\Delta_q \hat{a}_q^\dagger \hat{a}_q, \quad (\text{B9a})$$

$$\tilde{H}_m(t) = \Omega \hat{b}^\dagger \hat{b} - g_q |\alpha_q(t)|^2 \hat{x}^2, \quad (\text{B9b})$$

$$\tilde{H}_{qm}(t) = -g_q [\alpha_q^*(t) \hat{a}_q + \alpha_q(t) \hat{a}_q^\dagger] \hat{x}^2 - g_q \hat{a}_q^\dagger \hat{a}_q \hat{x}^2. \quad (\text{B9c})$$

The interest of moving to this picture is that now the driving has been moved to the coupling and the mechanical Hamiltonians, which will allow us to easily understand the physics behind the system. Moreover, in the absence of optomechanical coupling, the dynamics of the optical mode is generated by the simple Liouvillian $\mathcal{L}_q[\cdot] = i[\Delta_q \hat{a}_q^\dagger \hat{a}_q, \cdot] + \gamma_q \mathcal{D}_{a_q}[\cdot]$, including only detuning and dissipation, which drives it to a vacuum state at rate γ_q —which in the original picture corresponds to a coherent state with amplitude $\alpha_q(t)$.

Note finally that we could move to an even more accurate picture in which not only the optical, but also the mechanical mode are displaced to some reference phase-space point (e.g., the one corresponding to their configuration in the classical limit, see Appendix C). However, while it might be valuable from a quantitative point of view, this level of approximation is not required to understand the physics of the problem and identify the conditions required for the system to implement the DPO model, which is clearer in the simple optically displaced picture introduced above.

2. Derivation of effective mechanical master equation

Let us rewrite the master equation (B7) as

$$\frac{d\tilde{\rho}}{dt} = \mathcal{L}_m^{(t)}[\tilde{\rho}] + \mathcal{L}_q[\tilde{\rho}] + \mathcal{L}_{qm}^{(t)}[\tilde{\rho}], \quad (\text{B10})$$

where \mathcal{L}_q is defined above, while $\mathcal{L}_m^{(t)}[\cdot] = -i[\tilde{H}_m(t), \cdot] + \gamma_{\text{eff}} \mathcal{D}_b[\cdot]$ and $\mathcal{L}_{qm}^{(t)}[\cdot] = -i[\tilde{H}_{qm}(t), \cdot]$. Adiabatic elimination proceeds by choosing some reference state and dynamics for the optical mode, which is assumed to remain unperturbed by the mechanical mode, and hence the accuracy of the elimination depends crucially on the choice of a proper reference. For our purposes, it is enough to take the dynamics generated by \mathcal{L}_q as the optical reference, and hence its steady state $\tilde{\rho}_q = |0\rangle_q \langle 0|$ (vacuum) as the reference state. Let us then define the projector superoperator

$$\mathcal{P}[\cdot] = \tilde{\rho}_q \otimes \text{tr}_q\{\cdot\}, \quad (\text{B11})$$

and its complement $\mathcal{Q} = 1 - \mathcal{P}$. These superoperators satisfy the useful relations $\mathcal{P}\mathcal{L}_m^{(t)}[\cdot] = \mathcal{L}_m^{(t)}\mathcal{P}[\cdot]$ (obvious since $\mathcal{L}_m^{(t)}$ acts on the mechanics only) and $\mathcal{P}\mathcal{L}_q[\cdot] = 0 = \mathcal{L}_q\mathcal{P}[\cdot]$ (where the second equality is again obvious, while the first one comes from \mathcal{L}_q being traceless by conservation of probability).

The next step consists of projecting the master equation onto the corresponding subspaces defined by these superoperators. Defining the projected components of the density operator $\hat{u}(t) = \mathcal{P}[\tilde{\rho}(t)]$ and $\hat{w}(t) = \mathcal{Q}[\tilde{\rho}(t)]$, and using the properties of the projectors, it is straightforward to get the coupled linear

system

$$\frac{d\hat{u}}{dt} = (\mathcal{L}_m^{(t)} + \mathcal{P}\mathcal{L}_{qm}^{(t)})[\hat{u}] + \mathcal{P}\mathcal{L}_{qm}^{(t)}[\hat{w}], \quad (\text{B12a})$$

$$\frac{d\hat{w}}{dt} = (\mathcal{L}_m^{(t)} + \mathcal{L}_q + \mathcal{Q}\mathcal{L}_{qm}^{(t)})[\hat{w}] + \mathcal{Q}\mathcal{L}_{qm}^{(t)}[\hat{u}]. \quad (\text{B12b})$$

The second equation can be formally integrated, leading to

$$\hat{w}(t) = \int_0^t dt' \mathcal{T} \left\{ \exp \left[\int_{t'}^t dt'' (\mathcal{L}_m^{(t'')} + \mathcal{L}_q + \mathcal{Q}\mathcal{L}_{qm}^{(t'')}) \right] \right\} \times \mathcal{Q}\mathcal{L}_{qm}^{(t')}[\hat{u}(t')], \quad (\text{B13})$$

where \mathcal{T} is the time-ordering superoperator, and we have not written the term which depends on the initial value $\hat{w}(0)$ since in this concrete dissipative scenario any information related to it has to be completely washed out asymptotically. Next, we introduce this formal solution in the first equation and take the Born approximation in which we neglect terms beyond quadratic order in the interaction \mathcal{L}_{qm} , which allows us to write

$$\frac{d\hat{u}}{dt} = \mathcal{L}_m^{(t)}[\hat{u}] + \mathcal{P}\mathcal{L}_{qm}^{(t)}[\hat{u}] + \int_0^t d\tau \mathcal{P}\mathcal{L}_{qm}^{(t)} \mathcal{U}_m^{(t-\tau, t)} e^{\mathcal{L}_q \tau} \mathcal{Q}\mathcal{L}_{qm}^{(t-\tau)}[\hat{u}(t-\tau)], \quad (\text{B14})$$

where in addition we have made the variable change $t' = t - \tau$ in the integral, used the fact that \mathcal{L}_m and \mathcal{L}_q commute, and defined the mechanical time-evolution superoperator

$$\mathcal{U}_m^{(t-\tau, t)} = \mathcal{T} \{ e^{\int_{t-\tau}^t dt' \mathcal{L}_m^{(t')}} \}. \quad (\text{B15})$$

After performing the partial trace over the optical mode, this equation provides an effective master equation for the reduced mechanical state $\hat{\rho}_m = \text{tr}_q\{\tilde{\rho}\}$. In order to simplify further such equation, we need to write the explicit form of the interaction \mathcal{L}_{qm} , which we do as

$$\mathcal{L}_{qm}^{(t)}[\cdot] = i \sum_{j=1}^3 G_j(t) [\hat{B}_j \otimes \hat{x}^2, \cdot], \quad (\text{B16})$$

with

$$\hat{\mathbf{B}} = (\hat{a}_q, \hat{a}_q^\dagger, \hat{a}_q^\dagger \hat{a}_q), \quad (\text{B17a})$$

$$\mathbf{G}(t) = g_q [\alpha_q^*(t), \alpha_q(t), 1]. \quad (\text{B17b})$$

Note the null asymptotic expectation value of the optical operators, $\text{tr}_q\{\hat{B}_j \tilde{\rho}_q\} = 0 \forall j$, meaning that the second term of Eq. (B14) does not contribute since $\mathcal{P}\mathcal{L}_{qm}^{(t)} = 0$. Let us then define the optical correlation functions

$$\text{tr}_q\{\hat{B}_l e^{\mathcal{L}_q \tau} [\tilde{\rho}_q \hat{B}_j]\} = \lim_{t \rightarrow \infty} \langle \hat{B}_l(t) \hat{B}_j(t + \tau) \rangle_q \equiv K_{jl}(\tau), \quad (\text{B18a})$$

$$\text{tr}_q\{\hat{B}_l e^{\mathcal{L}_q \tau} [\hat{B}_j \tilde{\rho}_q]\} = \lim_{t \rightarrow \infty} \langle \hat{B}_l(t + \tau) \hat{B}_j(t) \rangle_q \equiv H_{jl}(\tau), \quad (\text{B18b})$$

where the expectation value refers to the picture rotating at the laser frequency and we have used the quantum regression theorem [117]. Then, the effective mechanical master equation

can be written as

$$\begin{aligned} \frac{d\hat{\rho}_m(t)}{dt} &\approx \mathcal{L}_m^{(t)}[\hat{\rho}_m] + \sum_{jl=1}^3 G_j(t)G_l(t) \int_0^t d\tau K_{jl}(\tau)[\hat{x}^2, \mathcal{U}_m^{(t-\tau,t)}[\hat{\rho}_m(t-\tau)\hat{x}^2]] \\ &+ \sum_{jl=1}^3 G_j(t)G_l(t) \int_0^t d\tau H_{jl}(\tau)[\mathcal{U}_m^{(t-\tau,t)}[\hat{x}^2\hat{\rho}_m(t-\tau)], \hat{x}^2], \end{aligned} \quad (\text{B19})$$

where, in order to simplify the equation and upcoming expressions, we neglected some terms coming from the time dependence of the couplings \mathbf{G} , which finds justification from conditions that will naturally appear later.

The correlation functions $K_{jl}(\tau)$ and $H_{jl}(\tau)$ can be evaluated in many ways. Instead of using the dynamics induced by \mathcal{L}_q in the Schrödinger picture [118], a particularly simple way of evaluating them is by using the equivalent quantum Langevin equations of the optical operators, which in the simple case of having dissipation and detuning only, consist of a single closed equation for the annihilation operator [115]:

$$\frac{d\hat{a}_q}{dt} = -(\gamma_q - i\Delta_q)\hat{a}_q + \sqrt{2\gamma_q}\hat{a}_{\text{in}}(t), \quad (\text{B20})$$

where the only nonzero input-operator correlators up to fourth order are

$$\begin{aligned} \langle \hat{a}_{\text{in}}(t_1)\hat{a}_{\text{in}}^\dagger(t_2) \rangle &= \delta(t_1 - t_2), \\ \langle \hat{a}_{\text{in}}(t_1)\hat{a}_{\text{in}}^\dagger(t_2)\hat{a}_{\text{in}}(t_3)\hat{a}_{\text{in}}^\dagger(t_4) \rangle &= \delta(t_1 - t_2)\delta(t_3 - t_4), \\ \langle \hat{a}_{\text{in}}(t_1)\hat{a}_{\text{in}}(t_2)\hat{a}_{\text{in}}^\dagger(t_3)\hat{a}_{\text{in}}^\dagger(t_4) \rangle &= \delta(t_1 - t_3)\delta(t_2 - t_4) \\ &+ \delta(t_1 - t_4)\delta(t_2 - t_3). \end{aligned} \quad (\text{B21})$$

The asymptotic ($t \gg \gamma_q^{-1}$) solution of this equation reads

$$\hat{a}_q(t) = \sqrt{2\gamma_q} \int_0^t dt' e^{-(\gamma_q - i\Delta_q)(t-t')} \hat{a}_{\text{in}}(t'), \quad (\text{B22})$$

which together with the correlators of the input operator allows us to write

$$K_{jl}(\tau) = e^{-(\gamma_q + i\Delta_q)\tau} \delta_{j1} \delta_{l2}, \quad (\text{B23a})$$

$$H_{jl}(\tau) = e^{-(\gamma_q - i\Delta_q)\tau} \delta_{j2} \delta_{l1}, \quad (\text{B23b})$$

which are functions decaying at rate γ_q .

Hence, of the dynamics induced by the time-evolution superoperator $\mathcal{U}_m^{(t-\tau,t)}$, we see that only the processes happening at a rate faster than or similar to γ_q play a role in the integral terms of the effective mechanical master equation (B19). This brings us to the final major approximation, known as the Markov approximation: we assume that of all the processes contributing to the mechanical dynamics, the only term acting appreciably on the timescale of the optical decay is the simple oscillation induced by the term $[\Omega - 2g_q(\bar{n}_0 + \bar{n}_1)]\hat{b}^\dagger\hat{b}$ of \hat{H}_m (we will justify this approximation self-consistently at the end of the derivation). Within this Markov approximation, we can then approximate

$$\begin{aligned} \mathcal{U}_m^{(t-\tau,t)}[\hat{\rho}_m(t-\tau)\hat{x}^2] &\approx e^{-i\Omega_{\text{eff}}\tau}\hat{b}^\dagger\hat{b}[\hat{\rho}_m(t-\tau)\hat{x}^2]e^{i\Omega_{\text{eff}}\tau}\hat{b}\hat{b}^\dagger \\ &\approx \hat{\rho}_m(t)\hat{x}(\tau)^2, \end{aligned} \quad (\text{B24})$$

where $\Omega_{\text{eff}} = \Omega - 2g_q(\bar{n}_0 + \bar{n}_1)$ and

$$\hat{x}(\tau) = e^{i\Omega_{\text{eff}}\tau}\hat{b} + e^{-i\Omega_{\text{eff}}\tau}\hat{b}^\dagger. \quad (\text{B25})$$

Similarly, we can approximate

$$\mathcal{U}_m^{(t-\tau,t)}[\hat{x}^2\hat{\rho}_m(t-\tau)] \approx \hat{x}(\tau)^2\hat{\rho}_m(t). \quad (\text{B26})$$

These approximations lead to the effective mechanical master equation

$$\frac{d\hat{\rho}_m}{dt} = \mathcal{L}_m^{(t)}[\hat{\rho}_m(t)] + [\hat{\Gamma}(t)\hat{\rho}_m - \hat{\rho}_m\hat{\Gamma}^\dagger(t), \hat{x}^2], \quad (\text{B27})$$

where, after performing the time integrals (in the asymptotic limit), it naturally appears the operator

$$\hat{\Gamma}(t) = \Gamma(t; \Omega_{\text{eff}})\hat{b}^2 + \Gamma(t; -\Omega_{\text{eff}})\hat{b}^{\dagger 2} + \Gamma(t; 0)(2\hat{b}^\dagger\hat{b} + 1), \quad (\text{B28})$$

with asymptotic time-dependent rates

$$\begin{aligned} \Gamma(t; \omega) &= \frac{\gamma_m C_q}{1 - i(\Delta_q + 2\omega)/\gamma_q} (1 + i\sqrt{\bar{n}_1/\bar{n}_0}e^{i\Omega_q t}) \\ &+ \frac{\gamma_m C_q}{1 - i(\Delta_q + \Omega_q + 2\omega)/\gamma_q} \\ &\times (\bar{n}_1/\bar{n}_0 - i\sqrt{\bar{n}_1/\bar{n}_0}e^{-i\Omega_q t}), \end{aligned} \quad (\text{B29})$$

where the cooperativity is defined as $C_q = g_q^2\bar{n}_0/\gamma_q\gamma_m$.

In order to see that this master equation has all the ingredients that we need plus many more, so it is just a matter of finding the regime in which the latter do not contribute, let us rewrite it. By defining the real and imaginary parts of the rates, $\Gamma_R(t; \omega) = \text{Re}\{\Gamma(t; \omega)\}$ and $\Gamma_I(t; \omega) = \text{Im}\{\Gamma(t; \omega)\}$ and defining the phonon-number operator $\hat{n} = \hat{b}^\dagger\hat{b}$, we can write

$$\begin{aligned} \frac{d\hat{\rho}_m}{dt} &= -i[\hat{H}_{\text{eff}}(t), \hat{\rho}_m] + \gamma_{\text{eff}}\mathcal{D}_b[\hat{\rho}_m] + \Gamma_R(t; \Omega_{\text{eff}})\mathcal{D}_{b^2}[\hat{\rho}_m] \\ &+ \Gamma_R(t; -\Omega_{\text{eff}})\mathcal{D}_{b^{\dagger 2}}[\hat{\rho}_m] + 4\Gamma_R(t; 0)\mathcal{D}_n[\hat{\rho}_m] \\ &+ \mathcal{L}_{\text{NRW}}^{(t)}[\hat{\rho}_m], \end{aligned}$$

where we have defined the effective Hamiltonian

$$\hat{H}_{\text{eff}}(t) = \hat{H}_{\text{DPO}}(t) + \hat{H}_{\perp\text{DPO}}(t), \quad (\text{B30})$$

containing terms that we will need for the DPO model

$$\hat{H}_{\text{DPO}}(t) = \Omega_{\text{eff}}\hat{n} + ig_q\sqrt{\bar{n}_0\bar{n}_1}(e^{-i\Omega_q t}\hat{b}^{\dagger 2} - e^{i\Omega_q t}\hat{b}^2), \quad (\text{B31})$$

plus some that we do not want to contribute

$$\begin{aligned} \hat{H}_{\perp\text{DPO}}(t) &= [4g_q\sqrt{\bar{n}_0\bar{n}_1}\sin(\Omega_q t)]\hat{n} - [\Gamma_I(t; \Omega_{\text{eff}}) \\ &- 3\Gamma_I(t; -\Omega_{\text{eff}}) - 4\Gamma_I(t; 0)]\hat{n} + [\Gamma_I(t; \Omega_{\text{eff}}) \\ &+ \Gamma_I(t; -\Omega_{\text{eff}}) + 4\Gamma_I(t; 0)]\hat{n}^2 - [g_q(\bar{n}_0 + \bar{n}_1) \\ &+ i\sqrt{\bar{n}_0\bar{n}_1}e^{i\Omega_q t}]\hat{b}^{\dagger 2} + \text{H.c.}], \end{aligned} \quad (\text{B32})$$

and we have collected into \mathcal{L}_{NRW} the dissipative terms which in the absence of the sideband are expected not to contribute within the rotating wave approximation, which read

$$\begin{aligned} \mathcal{L}_{\text{NRW}}^{(t)}[\hat{\rho}_m] = & [\Gamma(t; \Omega_{\text{eff}}) + \Gamma^*(t; -\Omega_{\text{eff}})]\hat{b}^2\hat{\rho}_m\hat{b}^2 \\ & - \Gamma(t; \Omega_{\text{eff}})\hat{b}^4\hat{\rho}_m - \Gamma^*(t; -\Omega_{\text{eff}})\hat{\rho}_m\hat{b}^4 \\ & + [\Gamma(t; \Omega_{\text{eff}}) + \Gamma^*(t; 0)]\hat{b}^2\hat{\rho}_m(2\hat{n} + 1) \\ & - \Gamma(t; \Omega_{\text{eff}})(2\hat{n} + 1)\hat{b}^2\hat{\rho}_m \\ & - \Gamma^*(t; 0)\hat{\rho}_m(2\hat{n} + 1)\hat{b}^2 \\ & + [\Gamma(t; 0) + \Gamma^*(t; -\Omega_{\text{eff}})](2\hat{n} + 1)\hat{\rho}_m\hat{b}^2 \\ & - \Gamma(t; 0)\hat{b}^2(2\hat{n} + 1)\hat{\rho}_m \\ & - \Gamma^*(t; -\Omega_{\text{eff}})\hat{\rho}_m\hat{b}^2(2\hat{n} + 1) + \text{H.c.} \quad (\text{B33}) \end{aligned}$$

3. Degenerate parametric oscillation regime

Let us now discuss the conditions under which the effective mechanical master equation above will correspond to the master equation of the DPO; Eq. (1).

Looking at the term $\hat{H}_{\text{DPO}}(t)$ of the effective Hamiltonian, we see that the sideband Ω_q should be chosen to match twice the effective mechanical frequency; that is, $2\Omega_{\text{eff}}$.

On the other hand, we would like the effective two-phonon cooling \mathcal{D}_{b^2} to dominate over any other irreversible process, in particular over the two-phonon heating $\mathcal{D}_{b^{i2}}$ and dephasing \mathcal{D}_n , and to do so with a time-independent rate. Looking at Eq. (B29), the latter can be naturally accomplished by driving the fundamental tone much stronger than the sideband, that is, $\bar{n}_0 \gg \bar{n}_1$. The static part of the rates (B29) then suggests that cooling will be enhanced by choosing a detuning of the fundamental driving tone matching the red two-phonon sideband, $\Delta_q = -2\Omega_{\text{eff}}$; indeed, this choice provides the following real, static part of the rates:

$$\Gamma_{\text{R}}(t; \Omega_{\text{eff}}) \rightarrow \gamma_m C_q \equiv \Gamma_{\text{cooling}}, \quad (\text{B34a})$$

$$\Gamma_{\text{R}}(t; -\Omega_{\text{eff}}) \rightarrow \frac{\gamma_m C_q}{1 + 16\Omega_{\text{eff}}^2/\gamma_q^2} \equiv \Gamma_{\text{heating}}, \quad (\text{B34b})$$

$$\Gamma_{\text{R}}(t; 0) \rightarrow \frac{\gamma_m C_q}{1 + 4\Omega_{\text{eff}}^2/\gamma_q^2} \equiv \Gamma_{\text{dephasing}}, \quad (\text{B34c})$$

showing in addition that we need to work in the resolved sideband regime $4\Omega_{\text{eff}}^2 \gg \gamma_q^2$ in order for heating and dephasing to be suppressed; in particular, we will define the parameter $r = (1 + 4\Omega_{\text{eff}}^2/\gamma_q^2)^{-1} \ll 1$, which allows us to approximate the rates by

$$\begin{aligned} \Gamma(t; \Omega_{\text{eff}}) = & \gamma_m C_q (1 + i\sqrt{r}\bar{n}_1/\bar{n}_0 + i\sqrt{\bar{n}_1/\bar{n}_0}e^{2i\Omega_{\text{eff}}t} \\ & + \sqrt{r(\bar{n}_1/\bar{n}_0)}e^{-2i\Omega_{\text{eff}}t}), \quad (\text{B35a}) \end{aligned}$$

$$\begin{aligned} \Gamma(t; -\Omega_{\text{eff}}) = & -i\sqrt{r}\gamma_m C_q (1 + i\sqrt{\bar{n}_1/\bar{n}_0}e^{2i\Omega_{\text{eff}}t} \\ & - 2i\sqrt{\bar{n}_1/\bar{n}_0}e^{-2i\Omega_{\text{eff}}t})/2, \quad (\text{B35b}) \end{aligned}$$

$$\begin{aligned} \Gamma(t; 0) = & \gamma_m C_q (\bar{n}_1/\bar{n}_0 - i\sqrt{r} + \sqrt{r(\bar{n}_1/\bar{n}_0)}e^{2i\Omega_{\text{eff}}t} \\ & - i\sqrt{\bar{n}_1/\bar{n}_0}e^{-2i\Omega_{\text{eff}}t}), \quad (\text{B35c}) \end{aligned}$$

expressions that together with working in the weak sideband regime, $\bar{n}_1/\bar{n}_0 \ll 1$, suggest that the only relevant rate is the

real static part of $\Gamma(t; \Omega_{\text{eff}})$, which provides the two-phonon cooling rate as desired.

The next constrain on the parameters comes from the fact that there are many counter-rotating terms that we do not want to contribute within the rotating-wave approximation. Among these, inspection of $\mathcal{L}_{\text{NRW}}^{(t)}$ and \hat{H}_{LDPO} shows that the largest of such rates are $\gamma_m C_q$ and $g_q(\bar{n}_0 + \bar{n}_1) \approx g_q\bar{n}_0$, but note that $g_q\bar{n}_0/\gamma_m C_q = \gamma_q/g_q$ which is typically much larger than 1 (optomechanical systems work far from the single-photon strong-coupling regime, specially when referring to the quadratic coupling), and hence $g_q\bar{n}_0$ is the largest of these two rates. Thus, the rotating wave approximation requires $g_q\bar{n}_0 \ll \Omega_{\text{eff}}$. Provided that this approximation holds, we can neglect all the counter-rotating terms in \hat{H}_{LDPO} and $\mathcal{L}_{\text{NRW}}^{(t)}$, approximating them by

$$\hat{H}_{\text{LDPO}} \approx -\sqrt{r}\gamma_m C_q(11 + 9\hat{n})\hat{n}/2, \quad (\text{B36})$$

and

$$\begin{aligned} \mathcal{L}_{\text{NRW}}^{(t)}[\hat{\rho}_m] = & \sqrt{\bar{n}_1/\bar{n}_0}\gamma_m C_q e^{2i\Omega_{\text{eff}}t} [i\hat{b}^2\hat{\rho}_m(2\hat{n} + 1) \\ & - i(2\hat{n} + 1)\hat{b}^2\hat{\rho}_m + \sqrt{r}\hat{\rho}_m(2\hat{n} + 1)\hat{b}^2 \\ & - \sqrt{r}\hat{b}^2(2\hat{n} + 1)\hat{\rho}_m + \sqrt{r}\hat{\rho}_m\hat{b}^2(2\hat{n} + 1)] + \text{H.c.} \quad (\text{B37}) \end{aligned}$$

This expression clearly shows that $\mathcal{L}_{\text{NRW}}^{(t)}$ is negligible when compared with the two-phonon-cooling term $\gamma_m C_q \mathcal{D}_{b^2}$. On the other hand, \hat{H}_{LDPO} provides a negligible effective mechanical frequency shift, but also a Kerr term which is expected to be negligible only as long as $\langle \hat{n} \rangle \ll \Omega_{\text{eff}}/4.5\sqrt{r}\gamma_m C_q$, which puts a bound on the number of phonons. Nevertheless, for the parameters corresponding to a realistic implementation used in the main text, we find this bound to be $\sim 10^{13}$, while the classical limit of the DPO tells us that the phonon number expected at $\sigma = 2.5$ is on the order of 10^{10} , three orders of magnitude below the limit in which the Kerr term can start playing a role.

Provided that all these considerations are taken into account, we then expect the effective mechanical master equation to be very well approximated by

$$\frac{d\hat{\rho}_m}{dt} \approx -i[\hat{H}_{\text{eff}}(t), \hat{\rho}_m] + \gamma_{\text{eff}}\mathcal{D}_b[\hat{\rho}_m] + \gamma_m C_q \mathcal{D}_{b^2}[\hat{\rho}_m], \quad (\text{B38})$$

with effective Hamiltonian

$$\hat{H}_{\text{eff}}(t) \approx \Omega_{\text{eff}}\hat{n} + ig_q\sqrt{\bar{n}_0\bar{n}_1}(e^{-2i\Omega_{\text{eff}}t}\hat{b}^{\dagger 2} - e^{2i\Omega_{\text{eff}}t}\hat{b}^2), \quad (\text{B39})$$

which is exactly the DPO model [Eq. (1)].

Note that there is a final constrain on the parameters coming from the Markov approximation that we performed in the adiabatic elimination: since we assumed that within the decay of the optical correlators at rate γ_q the only relevant mechanical process is the simple oscillation at frequency Ω_{eff} , we need the rates γ_{eff} , $\gamma_m C_q$, and $g_q\sqrt{\bar{n}_0\bar{n}_1}$ to be smaller than γ_q . For the parameters considered in the main text, γ_{eff} is the largest rate of the three (for an \bar{n}_1 corresponding to $\sigma = 2.5$ in the DPO model or smaller), and it satisfies $\gamma_{\text{eff}}/\gamma_q \approx 6 \times 10^{-3}$, so we are safely within the Markov regime.

APPENDIX C: ASYMPTOTIC STATE OF FULL MODEL

Let us in this final Appendix explain how we found the asymptotic state of the optomechanical model numerically. As with the stationary state of the DPO model, we performed simulations of the full master equation (B1), as well as simulations in the classical limit.

In the first case, our starting point has been the optomechanical master equation in the displaced picture; Eq. (B7). For numerical purposes, it is important to work in this displaced picture because the state of the optical mode should stay close to vacuum, while in the original picture it is a highly populated coherent state which does not allow for a reasonable truncation of the optical Fock space. As before, we set the truncation of the mechanical and optical Fock bases in such a way that the mean phonon number finds convergence up to the third significant digit, which typically does not require more than one or two photons in this displaced picture. The simulation proceeds again as explained in detail in Ref. [119], that is, by moving to superspace where the master equation (B7) is turned into a linear system $d\vec{\rho}(t)/dt = \mathbb{L}(t)\vec{\rho}(t)$. Note that now the linear problem is manifestly time dependent and, therefore, there will be no steady state. In particular, $\mathbb{L}(t)$ is $2\pi/\Omega_q$ periodic, and this periodicity is reflected in a time-dependent asymptotic state $\bar{\rho}(t) = \lim_{t \rightarrow \infty} \hat{\rho}(t)$, which we find by solving the linear system numerically starting from different initial conditions $\bar{\rho}(0)$; that is, different initial states $\hat{\rho}(0)$. In all the simulations we have checked that the asymptotic state is independent of the chosen initial state (e.g., vacuum or the steady state of the DPO for the mechanics). As explained in the text, the observable we have focused on is the asymptotic phonon number $\lim_{t \rightarrow \infty} \langle \hat{b}^\dagger \hat{b}(t) \rangle = \text{tr}\{\hat{b}^\dagger \hat{b} \bar{\rho}(t)\}$, whose time evolution can be approximated by a function of the type $\bar{n} + \delta n \sin(\Omega_q t)$, with $\delta n \ll \bar{n}$.

The superspace simulation of the master equation becomes quite heavy as the mechanical state gets populated, which has prevented us from performing simulations for values of the

sideband power where the system is expected to be above the DPO phase transition. Hence, in order to prove that the optomechanical model leads to the expected phase transition, we performed simulations of the optomechanical system in the classical limit. Similarly to what we did for the DPO model in the first section, this limit is found by assuming that both the optical and mechanical modes are in a coherent state at all times. Let us show the procedure explicitly for this case too. Our starting point is the original optomechanical master equation (B1), but replacing the mechanical dissipator $\mathcal{D}_b[\cdot]$ by $[\hat{x}, \{\hat{p}, \cdot\}]/2i$, where $\hat{p} = i(\hat{b}^\dagger - \hat{b})$ is the mechanical momentum quadrature. For high- Q mechanical oscillators which admit a weak-coupling description of their interaction with the environment, this dissipator leads to the same physics as the previous one [115], but provides better-looking classical equations. With this change, the evolution of the expectation value of any system operator \hat{A} reads

$$\begin{aligned} \frac{d\langle \hat{A} \rangle}{dt} = \text{tr} \left\{ \hat{A} \frac{d\hat{\rho}}{dt} \right\} = & -i \langle [\hat{A}, \hat{H}_m + \hat{H}_q(t) + \hat{H}_{qm}] \rangle \\ & + \gamma_q (\langle [\hat{a}_q^\dagger, \hat{A}] \hat{a}_q \rangle + \langle \hat{a}_q^\dagger [\hat{A}, \hat{a}_q] \rangle) + \frac{\gamma_q^{\text{eff}}}{2i} \langle \{[\hat{A}, \hat{x}], \hat{p}\} \rangle, \end{aligned} \quad (\text{C1})$$

where, just as with the DPO model, we are defining the expectation value with respect to the state in the picture rotating at the laser frequency; that is, $\langle \cdot \rangle = \text{tr}\{\cdot \hat{\rho}\}$, with $\hat{\rho}$ the state in the rotating frame. Applied to \hat{a}_q , \hat{x} , and \hat{p} , and denoting by $\alpha(t)$ and $\beta(t) = [x(t) + ip(t)]/2$ the amplitudes of the optical and mechanical coherent states, we find the classical evolution equations (8) that we introduced in Sec. V. Such a nonlinear system can be efficiently simulated numerically for (practically) any parameter set, and the phonon number that it predicts can be evaluated as $\lim_{t \rightarrow \infty} \langle \hat{b}^\dagger \hat{b}(t) \rangle = \lim_{t \rightarrow \infty} [x^2(t) + p^2(t)]/4$, which again can be approximated by $\bar{n} + \delta n \sin(\Omega_q t)$, with (typically) $\delta n \ll \bar{n}$.

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