Noncritically Squeezed Light via Spontaneous Rotational Symmetry Breaking

Carlos Navarrete–Benlloch, Eugenio Roldán, and Germán J. de Valcárcel

Departament d'Òptica, Universitat de València, Dr. Moliner 50, 46100–Burjassot, Spain

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We predict squeezed light generation through the spontaneous rotational symmetry breaking occurring in a degenerate optical parametric oscillator (DOPO) pumped above threshold. We show, within the linearized theory, that a DOPO with spherical mirrors, in which the signal and idler fields correspond to first-order Laguerre-Gauss modes, produces a perfectly squeezed vacuum with the shape of a Hermite-Gauss mode. This occurs at any pumping level above threshold; hence, the phenomenon is noncritical. Imperfections of the rotational symmetry, due, e.g., to cavity anisotropy, are shown to have a small impact.

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Introduction.—Squeezed light is a central tool in several applications of physics, high-precision measurements [1] and quantum information with continuous variables [2] being probably the most outstanding. The quality of squeezing, i.e., how less noisy light is as compared with vacuum (which sets the so-called shot noise level) is a main concern for those applications as any fluctuation level limits their performance. Improving the quality and reliability of squeezing is thus an important goal.

The paradigmatic squeezing process is single-mode quadrature squeezing of an optical field via degenerate parametric down-conversion [3,4], a nonlinear process that converts a pump photon of frequency $2\omega_0$ into two photons of frequency ω_0. Although in this process perfect squeezing is achieved only when the pump power goes to infinity, there is a well-known technique for increasing the squeezing level that consists in confining the nonlinear interaction inside an optical cavity, dealing then with a degenerate optical parametric oscillator (DOPO). In DOPOs squeezing is ideally obtained at the oscillation threshold [5]: a critical phenomenon. DOPOs are nowadays customarily utilized as sources of squeezed light, reaching noise reductions as large as 10 dB below the shot noise level (90% of squeezing) [6,7].

Recently an alternative to producing squeezed light was proposed by some of us [8], based on the spontaneous translational symmetry breaking occurring in a broad area, planar DOPO model. Such a system supports cavity solitons (CSs) (among other dissipative structures forming across its transverse plane), which are localized light structures that can be placed at any point in space, thus breaking the translational symmetry. A study of their quantum fluctuations [8] reveals that (i) the CS position diffuses because of quantum noise, and (ii) a special transverse mode quadrature, namely, the $\frac{\pi}{2}$ phase shifted gradient of the CS (its linear momentum), is perfectly squeezed at low fluctuation frequencies, irrespective of the system's proximity to threshold. This is reminiscent of a Heisenberg uncertainty relation, with the additional and compatible feature that the full indetermination of the CS position in the long time limit is accompanied by the perfect determination (perfect squeezing) of its momentum at low frequencies, like a canonical pair in a minimum uncertainty state. A main limitation of this result is that CSs have not been observed so far in DOPOs. Nevertheless, it paves the way (noncritical squeezing via a spontaneous spatial symmetry breaking) to other extensions, like the one we consider here: the spontaneous rotational symmetry breaking of a DOPO that can be implemented with current technology. We hope that experiments based on this new phenomenon will successfully generate high-quality noncritically squeezed light.

Rotational invariance and squeezing: General description.—Consider a type I DOPO (in which signal and idler photons are degenerated in both frequency and polarization) with spherical mirrors and pumped by a resonant, coherent optical field of frequency $2\omega_0$ and Gaussian transverse profile [of zero orbital angular momentum (OAM)]. Such a configuration is invariant under rotations around the cavity axis (z axis). The cavity is assumed to be tuned to the first transverse mode family at the subharmonic frequency ω_0, so the signal (or idler) field is a superposition of two Laguerre-Gauss (LG) modes, $L_{+1}(\mathbf{r})$ and $L_{-1}(\mathbf{r})$, having opposite OAM. Any other transverse mode is assumed to be detuned far enough from ω_0. Inside this cavity a $\chi^{(2)}$ crystal down converts pump photons into signal-idler photon pairs, and vice versa. These photons are degenerate in frequency because of energy conservation, and owed to OAM conservation, each photon pair must comprise one L_{+1} photon plus one L_{-1} photon. Hence the number of L_{+1} photons and L_{-1} photons should be sensibly equal and highly correlated. Another way of looking at this process follows from noticing that the simultaneous emission of a L_{+1} photon and a L_{-1} photon corresponds to the emission of two photons in a first-order Hermite-Gauss (HG), or TEM_{10}, mode, which breaks the rotational symmetry of the system. The orientation of such mode in the transverse plane, measured by the angle θ in Fig. 1, is determined by the relative phase between the two subjacent LG modes. The rotational in-

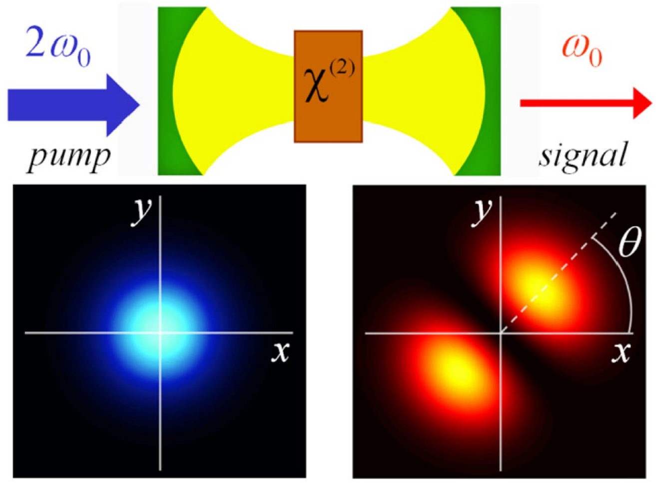

FIG. 1 (color online). Scheme of the DOPO pumped by a Gaussian beam and tuned to the first transverse mode family at the subharmonic. θ is arbitrary.

variance of the system implies, however, that θ is arbitrary and quantum fluctuations will induce a random rotation of the TEM_{10} mode around the cavity axis. In loose terms this means an "indefiniteness" in the value of θ that, generalizing Ref. [8], should be accompanied by a reduction of fluctuations in the canonically conjugated variable, the OAM, associated to the operator $-i\frac{\partial}{\partial\theta}$. We note that the angular gradient of the TEM_{10} mode is another HG mode, spatially crossed with respect to it, call it TEM_{01} mode. Hence a balanced homodyne detection that uses as a local oscillator a $\frac{\pi}{2}$ phase shifted TEM_{01} mode should yield perfect squeezing at zero noise frequency at any pumping level above threshold. This is the basic idea of squeezing generation via spontaneous rotational symmetry breaking in DOPO, and below we demonstrate that this is what actually occurs [9].

The model.—There are three relevant cavity modes: the pumped Gaussian mode at frequency $2\omega_0$, and two LG modes at the subharmonic frequency ω_0. The electric field at the cavity waist (where the nonlinear crystal is assumed to be located) can be written as

$$\hat{E}(\mathbf{r},t) = i\mathcal{F}_p\hat{A}_p(\mathbf{r},t)e^{-2i\omega_0 t} + i\mathcal{F}_s\hat{A}_s(\mathbf{r},t)e^{-i\omega_0 t} + \text{H.c.}, \tag{1}$$

where $\mathcal{F}_p = \sqrt{2}\mathcal{F}_s = \sqrt{2\hbar\omega_0/n\varepsilon_0 L}$, L is the effective cavity length, n is the crystal refractive index,

$$\hat{A}_p(\mathbf{r},t) = \hat{a}_0(t)G(\mathbf{r}), \tag{2a}$$

$$\hat{A}_s(\mathbf{r},t) = \hat{a}_{+1}(t)L_{+1}(\mathbf{r}) + \hat{a}_{-1}(t)L_{-1}(\mathbf{r}), \tag{2b}$$

are slowly varying envelopes, and $\hat{a}_m(t)$ and $\hat{a}_m^\dagger(t)$ are the interaction picture boson operators for each mode ($m = 0, \pm1$) obeying $[\hat{a}_m(t), \hat{a}_n^\dagger(t)] = \delta_{mn}$. The Gauss, $G(\mathbf{r})$, and LG, $L_{\pm1}(\mathbf{r})$, mode envelopes are given by [10] $G(\mathbf{r}) = \sqrt{2}\pi^{-1/2}w^{-1}e^{-r^2/w^2}$ and $L_{\pm1}(\mathbf{r}) = \pi^{-1/2}w^{-2}re^{-r^2/2w^2}e^{\pm i\phi}$, r and ϕ are the polar coordinates in the transverse plane, and w ($\sqrt{2}w$) is the beam radius of the pump (signal) beam at its waist.

The interaction Hamiltonian describing pumping and the nonlinear mixing processes occurring at the nonlinear crystal reads $\hat{H} = i\hbar(\mathcal{E}_p\hat{a}_0^\dagger + \chi\hat{a}_{+1}^\dagger\hat{a}_{-1}^\dagger\hat{a}_0) + \text{H.c.}$, where $\mathcal{E}_p$ is the amplitude of the external coherent pump, real without loss of generality, and χ is the nonlinear coupling constant [11]. Assuming that pump and signal modes are damped at rates γ_p and γ_s, respectively, losses occurring at only one cavity mirror (which needs not be the same for pump and signal), one can write down Langevin equations using well-known techniques of quantum optics of open systems. We notice, however, that our Hamiltonian is formally equivalent to that for the nondegenerate optical parametric oscillator and use the corresponding Langevin equations in the positive P representation as given in [12]. This representation sets a correspondence between operators $\{\hat{a}_m, \hat{a}_m^\dagger\}$ and independent c-number stochastic variables $\{\alpha_m, \alpha_m^+\}$, respectively, so that any stochastic average equals the corresponding normally ordered quantum expectation value. We further simplify the problem by considering the limit $\gamma_p \gg \gamma_s$, where the pump mode can be adiabatically eliminated as $\alpha_0 = (\mathcal{E}_\text{p} - \chi\alpha_{+1}\alpha_{-1})/\gamma_\text{p}$, arriving at our model equations:

$$\dot{\alpha}_i = \gamma_s(-\alpha_i + \sigma\alpha_j^+ - g^2\alpha_j^+\alpha_j\alpha_i) + \sqrt{\chi\alpha_0}\xi_i, \tag{3}$$

the overdot meaning d/dt, $i, j = \pm1$ ($i \neq j$), and the dimensionless parameters

$$\sigma = \mathcal{E}_p\chi/\gamma_p\gamma_s, \qquad g = \chi/\sqrt{\gamma_p\gamma_s}, \tag{4}$$

σ^2 being proportional to the external pump power. Another equation for $\dot{\alpha}_i^+$ exists that reads $\dot{\alpha}_i^+ = (\dot{\alpha}_i)^+$, where the operation "+" formally acts as a Hermitian conjugation. In Eq. (3), $\xi_{+1} = \xi_{-1}^* \equiv \xi$, $\xi_{+1}^+ = (\xi_{-1}^+)^* \equiv \xi^+$, and (ξ, ξ^+) are two independent complex white noises with zero mean and nonzero correlations $\langle\xi(t_1)\xi^*(t_2)\rangle = \langle\xi^+(t_1)[\xi^+(t_2)]^*\rangle = \delta(t_1 - t_2)$.

Classical steady emission.—The DOPO classical dynamical equations are obtained by setting $\alpha_{\pm1}^+ = \alpha_{\pm1}^*$ and ignoring noise terms in Eq. (3). Above threshold ($\sigma > 1$) the only stable steady state reads

$$\bar{\alpha}_{\pm1} = \rho\exp(\mp i\theta), \qquad \rho^2 = g^{-2}(\sigma - 1), \tag{5}$$

with θ an arbitrary phase. The corresponding classical slowly varying envelope is obtained from Eq. (2b) after the replacement $\{\hat{a}_m, \hat{a}_m^\dagger\} \to \{\bar{\alpha}_m, \bar{\alpha}_m^*\}$ and reads

$$A_s^{\text{cl}}(\mathbf{r}) = (2\pi^{-1/2}w^{-2}\rho)r\cos(\phi - \theta)e^{-r^2/2w^2}, \tag{6}$$

which is a first-order HG mode rotated by θ with respect to the transverse x axis; see Fig. 1. The arbitrariness of θ reflects the rotational invariance of the problem.

Quantum fluctuations.—The dynamics of quantum fluctuations around the classical solution is studied by writing

$\alpha_{\pm 1} = \bar{\alpha}_{\pm 1} + \delta\alpha_{\pm 1}$ and deriving evolution equations for the fluctuations $\delta\alpha$'s. As $|\bar{\alpha}_{\pm 1}|^2 \gg 1$ (these quantities give the classical number of signal photons in each mode, which are very large above threshold), we assume that $|\delta\alpha_{\pm 1}|$, $|\delta\alpha^+_{\pm 1}| \ll |\bar{\alpha}_{\pm 1}|$ and linearize Eq. (3). We find it convenient to write the fluctuations as $\delta\alpha_{\pm 1} = b_{\pm 1} e^{\mp i\theta}$, i.e.,

$$\alpha_{\pm 1}(t) = [\rho + b_{\pm 1}(t)] e^{\mp i\theta(t)}, \tag{7}$$

with $b_{\pm 1}$ (and $b^+_{\pm 1}$) c-number stochastic variables accounting for quantum fluctuations. Note that angle θ is allowed to vary with time as, owed to rotational invariance, it is an undamped quantity driven by quantum noise as we show below. Inserting (7) into (3) one easily gets the linearized Langevin equations

$$\dot{\mathbf{b}} - 2i\rho \mathbf{w}_0 \dot{\theta} = \mathcal{L}\mathbf{b} + \sqrt{\gamma_s}\boldsymbol{\xi}(t). \tag{8}$$

$\mathbf{b} = \mathrm{col}(b_{+1}, b^+_{+1}, b_{-1}, b^+_{-1})$, $\mathbf{w}_0 = \frac{1}{2}\mathrm{col}(1, -1, -1, 1)$, $\boldsymbol{\xi} = \mathrm{col}(\xi, \xi^*, \xi^+, [\xi^+]^*)$, and the real and symmetric matrix $\mathcal{L}$ reads

$$\mathcal{L} = -\gamma_s \begin{pmatrix} \sigma & 0 & \sigma - 1 & -1 \\ 0 & \sigma & -1 & \sigma - 1 \\ \sigma - 1 & -1 & \sigma & 0 \\ -1 & \sigma - 1 & 0 & \sigma \end{pmatrix}. \tag{9}$$

The eigensystem of $\mathcal{L}$ consists of the Goldstone mode $\mathbf{w}_0$, whose null eigenvalue reflects the rotational invariance of the system, of vector $\mathbf{w}_1 = \frac{1}{2}\mathrm{col}(-1, -1, 1, 1)$, with eigenvalue $-2\gamma_s$, and of two other eigenvectors that are unimportant for our present purposes.

Angular diffusion of the classical field.—In order to catch the dynamics of the pattern orientation angle θ (see Fig. 1), we project the linear system (8) onto the Goldstone mode $\mathbf{w}_0$ and obtain [13,14]

$$\dot{\theta} = \sqrt{D_\theta}\,\mathrm{Im}(\xi^+ - \xi), \qquad D_\theta = \chi^2/4\gamma_p(\sigma - 1). \tag{10}$$

Equation (10) describes a diffusion of θ as already advanced, leading to a variance $\langle[\theta(t) - \theta(0)]^2\rangle = D_\theta t$, with D_θ the diffusion coefficient. Using common values for the system parameters [15] we find $D_\theta \sim 10^{-6}\ \mathrm{s}^{-1}$ for a pump power twice above threshold ($\sigma^2 = 2$). Hence the rotation of the classical HG mode will be minute unless the system is terribly close to threshold. This occurs as the TEM_{10} mode is macroscopically occupied and hence presents strong inertia to rotations. This is a most relevant conclusion as a rapid, random rotation of the output TEM_{10} mode would entail practical difficulties for the homodyne detection we pass to describe.

Homodyne detection and squeezing spectrum.—In order to demonstrate that the signal field exiting the DOPO exhibits perfect squeezing (within the linear approach) in the empty HG mode perpendicular to the macroscopically emitted one, we consider a balanced homodyne detection experiment; see, e.g., [16]. The noise spectrum $V(\omega)$ of the intensity difference between the two output ports of the beam splitter, in which the signal field exiting the DOPO is mixed with a classical, coherent local oscillator (LO) of frequency ω_0, is given by [8,16]

$$V(\omega) = 1 + 2\gamma_s \int_{-\infty}^{+\infty} d\tau \langle \delta\mathcal{E}(t)\delta\mathcal{E}(t + \tau)\rangle e^{-i\omega\tau}, \tag{11}$$

where the positive P representation is used to evaluate the stochastic average, $\delta\mathcal{E}(t) = \mathcal{E}(t) - \langle\mathcal{E}(t)\rangle$ with

$$\mathcal{E}(t) = \mathcal{N}^{-1/2} \int d^2\mathbf{r}(A_L^* A_s + A_L A_s^+), \tag{12}$$

$\mathcal{N} = \int d^2\mathbf{r}|A_L|^2$, and an argument $(\mathbf{r}, t)$ should be understood in all fields. $A_L(\mathbf{r}, t)$ is the LO transverse envelope and $A_s(\mathbf{r}, t) = \alpha_{+1}(t)L_{+1}(\mathbf{r}) + \alpha_{-1}(t)L_{-1}(\mathbf{r})$. When the output is coherent, $V(\omega) = 1$ for all ω, defining the shot noise level. On the other hand, $V(\omega_*) = 0$ signals perfect squeezing (no noise) at $\omega = \omega_*$ for the quadrature selected by the LO.

Following the Introduction we choose the LO transverse envelope to be $A_L(\mathbf{r}, t) \propto \frac{\partial}{\partial\theta} A_s^{\mathrm{cl}}(\mathbf{r})$, i.e.,

$$A_L(\mathbf{r}, t) = \eta_L e^{i\psi_L} r \sin[\phi - \theta(t)] e^{-r^2/2w^2}, \tag{13}$$

with η_L the real amplitude and ψ_L the phase of the LO. This LO is a HG mode orthogonal, at every time, to the macroscopically excited one, Eq. (6). We recall that if the system is sufficiently above threshold the diffusion of θ will be negligible and the matching of the LO to the analyzed mode should not represent serious problems. Plugging (7) and (13) into (12) one finds $\delta\mathcal{E}(t) = \sqrt{2}\sin(\psi_L)c_1(t)$, with $c_1(t) = \mathbf{w}_1 \cdot \mathbf{b}(t)$. Projecting (8) onto $\mathbf{w}_1$ gives [13]

$$\dot{c}_1 = -2\gamma_s c_1 - i\sqrt{\gamma_s}\,\mathrm{Im}(\xi^+ + \xi), \tag{14}$$

which allows evaluating the squeezing spectrum (11),

$$V_{\psi_L}(\omega) = 1 - \frac{\sin^2(\psi_L)}{1 + (\omega/2\gamma_s)^2}, \tag{15}$$

which is a main result of this Letter. For $\psi_L = \frac{\pi}{2}$, Eq. (15) coincides exactly with the squeezing spectrum of a usual DOPO at threshold [5], displaying perfect squeezing ($V = 0$) at $\omega = 0$: The phase quadrature of the TEM_{01} mode orthogonal to the TEM_{10} emitted by the DOPO is perfectly squeezed at zero noise frequency. The remarkable difference with usual DOPOs is that the result here reported is independent of the system parameters (e.g., it is not sensitive to bifurcations): It is thus a noncritical phenomenon, or in other words, squeezing needs not be tuned.

Some extra comments are in order: (i) $V_{\psi_L}(\omega) \leq 1$ for any LO phase ψ_L, i.e., any quadrature exhibits noise reduction, but $\psi_L = 0$ for which $V_{\psi_L=0}(\omega) = 1$; (ii) $V_{\psi_L}(\omega)V_{\psi_L+(\pi/2)}(\omega) < 1$, i.e., any two orthogonal quadratures measured by the LO are not a Heisenberg pair, what looks surprising but is understood by the fact that the detected mode (and thus the LO) is rotating randomly [see [17] and the discussion after Eq. (17) below]; (iii) the detected squeezing is very weakly dependent on ψ_L, e.g., phase uncertainties of ψ_L, around $\psi_L = \frac{\pi}{2}$, as huge as 15° lead to $V(\omega = 0) \simeq 0.067$ (more than 11 dB of noise reduction), unlike conventional squeezers [6,7].

Influence of imperfections.—A natural question is whether the above result is singular: Can deviations from perfect rotational invariance destroy it? We address this issue by considering different cavity losses, γ_x and γ_y, along two orthogonal transverse directions. The corresponding Langevin equations read

$$\dot{\alpha}_i = \gamma_s(-\alpha_i + \kappa\alpha_j + \sigma\alpha_j^+ - g^2\alpha_j^+\alpha_j\alpha_i) + \sqrt{\gamma_x}\xi_i, \tag{16}$$

where noises have been written in the linear approximation to be used, $\gamma_s = \frac{\gamma_y+\gamma_x}{2}$, $\kappa = \frac{\gamma_y-\gamma_x}{\gamma_y+\gamma_x}$ measures how much the rotational symmetry is externally broken, and $\gamma_y > \gamma_x$ for definiteness ($0 < \kappa < 1$). Above threshold ($\sigma > 1 - \kappa$) the classical steady state is $\bar{\alpha}_{+1} = \bar{\alpha}_{-1} = g^{-1}\sqrt{\sigma + \kappa - 1}$, corresponding to a horizontal HG mode [18], parallel to the direction of smaller losses. Use of Eq. (7) (now with $\theta = 0$) and considering a vertical HG mode as the LO, one gets

$$V_{\psi_L=(\pi/2)}(\omega) = 1 - \frac{1-\kappa^2}{1+(\omega/2\gamma_s)^2}, \tag{17}$$

independently of the pump level, which reduces to (15) for $\kappa = 0$ ($\gamma_y = \gamma_x$). One can show easily that $V_{\psi_L=0}(\omega)V_{\psi_L=(\pi/2)}(\omega) = 1$, corresponding to a minimum uncertainty state, as usual in DOPOs: The apparent violation of the Heisenberg relation in the previous section was related to the detection scheme (now the LO is kept fixed). Note that $\kappa \neq 0$ does not destroy the squeezing phenomenon described before: For example, for $\kappa = \frac{1}{3}$ ($\gamma_y = 2\gamma_x$, a huge anisotropy indeed), $V_{\psi_L=(\pi/2)}(\omega = 0) \simeq 0.11$ ($\simeq$ 10 dB of noise reduction), a very large squeezing level. This simple approach suggests that the phenomenon presented here is very robust.

Concluding remarks.—We have shown that the spontaneous rotational symmetry breaking around the cavity axis in a type I DOPO above threshold leads to noncritically squeezed light in the form of a first-order Hermite-Gauss mode, orthogonal to the one in which bright emission occurs, what can be of utility for precision measurements [19]. Squeezing, perfect in the linear approach at zero noise frequency, is independent of the system's distance from threshold. The squeezed mode orientation diffuses very slowly, typically with a diffusion coefficient $D_\theta \sim 10^{-6}\ \text{s}^{-1}$. In a standard experiment using a fixed local oscillator this could lead to a very small mode mismatch during the homodyne detection, thus degrading the detected squeezing. Such possible problems are absent when rotational symmetry is imperfect, e.g., due to a cavity anisotropy, in which case very large squeezing levels are still attained, evidencing the robustness of the phenomenon.

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